B.Sc. (Math) part I paper II

The plane

Def:— The plane:— A plane is one of the simplest types of locus in the three-dimensional space

Theorem To prove that the general equation of first degree in x, y, z represents a plane.

proof:— Suppose the equation of first degree in x, y, z be

$$ax + by + cz + d = 0 \quad \text{—— (1)}$$

where a, b, c, d are any given constant, a, b, c being not all zero.

The locus of this eqn. will be a plane if every point in the plane joining any two points on the locus also lies on the locus.

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points on the locus of (1). Then $ax_1 + by_1 + cz_1 + d = 0$ —— (2)

and $ax_2 + by_2 + cz_2 + d = 0$ —— (3)

Multiply (3) by $K(\neq -1)$ and adding in (1) we get

$$a(x_1 + Kx_2) + b(y_1 + Ky_2) + c(z_1 + Kz_2) + d(1+K) = 0$$

Dividing by $(1+K)$ we get

$$a\left(\frac{x_1+Kx_2}{1+K}\right)+b\left(\frac{y_1+Ky_2}{1+K}\right)+c\left(\frac{z_1+Kz_2}{1+K}\right)+d=0$$

This shows that the point

$$\left(\frac{x_1+Kx_2}{1+K}, \frac{y_1+Ky_2}{1+K}, \frac{z_1+Kz_2}{1+K}\right)$$

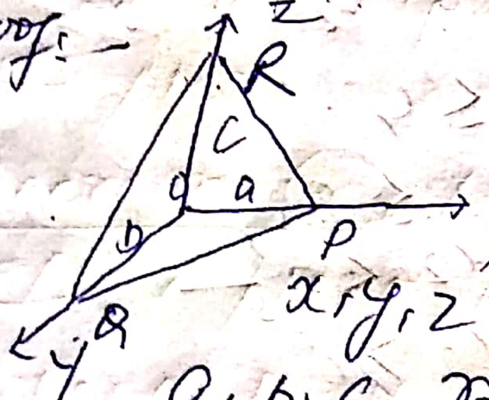
is also on the locus of ① for all values of K.

Thus if the points (x_1, y_1, z_1) (x_2, y_2, z_2) be on the locus, any other point in ~~plane~~ the line joining also lies on the locus. Hence the give equation represents a plane.

Thus every equation of first degree in x, y, z represents a plane.

Theorem:- To find the equation of a plane cutting off intercepts a, b, c from the axes.

proof:-



Let $Ax+By+Cz+D=0$ be eqn. of plane ①

It cuts the x, y, z axes at distance a, b, c respectively from origin. Thus $OP=a$, $OQ=b$, OR

then the Co-ordinate of P, Q and R are $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$.

Since the plane ① passes through the points $P(a,0,0)$, $Q(0,b,0)$ and $R(0,0,c)$ the Co-ordinates of these points must satisfy the equation ① of the plane

$\therefore A \cdot a + B \cdot 0 + C \cdot 0 + D = 0$ or $A = -\frac{D}{a}$

Similarly $B = -\frac{D}{b}$ and $C = -\frac{D}{c}$

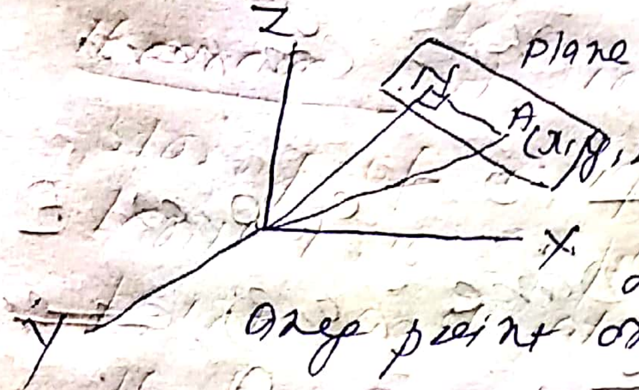
Substituting the values of A, B, and C in ① we get

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0$$

$$\Rightarrow \boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}$$

This is required equation of plane.

Theorem ③ To prove the equation of a plane is $lx + my + nz = p$ where p is the perpendicular distance from origin to the plane and l, m, n are the direction cosines of that perpendicular.



Let $ON = p$ and let l, m, n be the direction cosine of the normal ON. Let $A(x_1, y_1, z_1)$ be any point on plane.

Now the points A and N are in the plane, Therefor the line joining A and N must be in the plane. Again ON is normal to plane. Therefore ON must be perpendicular to NA.

The projection of OA on ON $=$ ON

$\therefore (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n = ON$

Here $x_1 = 0, y_1 = 0, z_1 = 0, x_2 = x, y_2 = y$, $z_2 = z$, $ON = p$

$\therefore (x-0)l + (y-0)m + (z-0)n = p$

$\therefore lx + my + zn = p$